

# Understanding Complex Networks Using Graph Spectrum

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## ABSTRACT

Complex networks are becoming indispensable parts of our lives. The Internet, wireless (cellular) networks, online social networks, and transportation networks are examples of some well-known complex networks around us. These networks generate an immense range of big data: weblogs, social media, the Internet traffic, which have increasingly drawn attentions from the computer science research community to explore and investigate the fundamental properties of, and improve the user experiences on, these complex networks. This work focuses on understanding complex networks based on the graph spectrum, namely, developing and applying spectral graph theories and models for understanding and employing versatile and oblivious network information – asymmetrical characteristics of the wireless transmission channels, multiplex social relations, e.g., trust and distrust relations, etc – in solving various application problems, such as estimating transmission cost in wireless networks, Internet traffic engineering, and social influence analysis in social networks.

## Categories and Subject Descriptors

G.2.2 [Graph Theory]: Network problems, Graph algorithms;  
H.3.3 [Information Systems]: Information Storage and Retrieval—*Information Search and Retrieval*

## General Terms

Theory

## Keywords

Spectral Graph Theory; Complex Network Analysis

## 1. INTRODUCTION

Complex networks arising from many applications can be represented and studied as graphs. For example, in an ad hoc wireless network, nodes represent wireless devices, whereas edges characterize the available wireless links among those devices. On an online social network (OSN), users and their social interactions can

be viewed as nodes and edges in a graph. The topological properties of the underlying complex networks play an important role in understanding and solving the application problems.

In mathematics, *spectral graph theory*, as an enabling tool, establishes relationship between various properties of a graph and the characteristic polynomial, eigenvalues, and eigenvectors of matrices associated to the graph, such as its adjacency matrix or Laplacian matrix. For examples, the number of zero eigenvalues of the graph Laplacian matrix indicates the number of disconnected components of the network. The pseudo-inverse of the graph Laplacian matrix infers the hitting time and commute time of a random walk on the underlying network. These spectral properties of the graphs can be utilized to tackle application problems in various seemingly unrelated network settings. In this paper, we summarize works in the literature, that develops and applies spectral graph theory to study the crucial and unique properties of various types of graphs, such as directed graphs (digraphs), undirected graphs, and signed graphs, with applications to wireless networking [11, 12, 15], the Internet traffic engineering [17, 18], and online social network (OSN) analysis [9, 16, 10].

To our best knowledge, this is the first work that provides comprehensive study on applications of spectral graph theory on various application problems. The paper is organized as follows. Section 2 provides an overview of the spectral graph theory. Section 3 discusses various applications of spectral graph theory under three graph models, including directed graphs, undirected graphs, and signed graphs. To be specific, Section 3.1 introduces a generalized digraph spectral graph theory [13, 14] with applications in estimating wireless transmission costs [12]. Section 3.2 studies the setting of undirected graph, with application of Internet traffic engineering. Section 3.3 discusses the generalization of spectral graph theory to signed graphs with both positive and negative links, and the application scenario is social influence maximization in online viral marketing. Section 4 discusses the future research directions, and Section 5 concludes the paper.

## 2. PRELIMINARIES

Emerging in the 1950s, *spectral graph theory* studies the properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvectors of matrices associated to the graph, such as its adjacency matrix or Laplacian matrix. Earlier studies primarily focus on undirected graphs [6, 20, 7, 5, 3, 4], with symmetric adjacency matrices, real eigenvalues, and a complete set of orthonormal eigenvectors. The entire set of eigenvalues and eigenvectors form the graph's spectrum.

We use a triple  $G = (V, E, A)$  to denote an *undirected* and *weighted* graph on the node set  $V = \{1, 2, \dots, n\}$ . The  $n \times n$  (nonnegative) weight matrix  $A = [a_{ij}]$  is *symmetric*, and is defined

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in such a way that  $a_{ij} = a_{ji} > 0$ , if  $(i, j) \in E$ , and  $a_{ij} = a_{ji} = 0$  otherwise. For  $1 \leq i \leq n$ , the degree of node  $i$  is  $d_i = \sum_{j=1}^n a_{ij}$ . The *volume* of  $G$ , denoted by  $vol(G)$ , is defined as the sum of all node degrees,  $d = \sum_{i=1}^n d_i$ , i.e.,  $vol(G) = d$ .

A *random walk* on  $G$  is a Markov chain defined on  $G$  with the transition probability matrix  $P = [p_{ij}]$ , where  $p_{ij} = a_{ij}/d_i$ . Let  $D = \text{diag}[d_i]$  be a diagonal matrix of node degrees, then  $P = D^{-1}A$ . Without loss of generality, we assume that the undirected graph  $G$  is connected (i.e., any node can reach any other node in  $G$ ). Then it can be shown (see, e.g., [1]) that the Markov chain is irreducible, and there exists a unique stationary distribution,  $\{\pi_1, \pi_2, \dots, \pi_n\}$ . Let  $\pi = [\pi_i]_{1 \leq i \leq n}$  be the *column* vector of the stationary probabilities. Then  $\pi^T P = \pi^T$ , where the superscript  $T$  represents (vector or matrix) transpose. Furthermore, this Markov chain (random walk) on  $G$  is *reversible*, namely

$$\pi_i p_{ij} = \pi_j p_{ji}, \text{ for any } i, j, \quad (1)$$

and

$$\pi_i = \frac{d_i}{\sum_k d_k} = \frac{d_i}{d}, \quad i = 1, 2, \dots, n. \quad (2)$$

[6] uses the *normalized* graph Laplacian (instead of the unnormalized version  $L = D - A$ ). Given an undirected  $G$ , the *normalized* graph Laplacian of  $G$  (also called normalized Laplacian matrix of  $G$ ) is defined as follows:

$$\mathcal{L} = D^{-\frac{1}{2}}(D - A)D^{-\frac{1}{2}} = D^{\frac{1}{2}}(I - P)D^{-\frac{1}{2}}. \quad (3)$$

A key property of the graph Laplacian (for an undirected graph) is that  $\mathcal{L}$  is symmetric and *positive semi-definite* [8]. Hence all eigenvalues of  $\mathcal{L}$  are *nonnegative real numbers*. In particular, for a connected undirected graph  $G$ ,  $\mathcal{L}$  has rank  $n-1$  and has exactly one zero eigenvalue (its smallest one). Let  $\lambda_1 = 0 < \lambda_2 \leq \dots \leq \lambda_n$  be the  $n$  eigenvalues of  $\mathcal{L}$  arranged in an increasing order, and  $\mu_i$ ,  $1 \leq i \leq n$ , be the corresponding eigenvectors (of unit norm). In particular, one can show that the (column) eigenvector,  $\mu_1$ , of  $\mathcal{L}$  associated with the eigenvalue  $\lambda_1 = 0$ , is given by

$$\mu_1 = \pi^{\frac{1}{2}} = [\sqrt{\pi_i}] = \left[ \frac{\sqrt{d_i}}{\sqrt{d}} \right]. \quad (4)$$

Define  $\Gamma := \text{diag}[\lambda_1, \dots, \lambda_n]$ , the diagonal matrix formed by the eigenvalues, and  $U = [\mu_1, \dots, \mu_n]$ , an orthonormal matrix formed by the eigenvectors of  $\mathcal{L}$ , where  $UU^T = U^T U = I$ . It is easy to see that the graph Laplacian  $\mathcal{L}$  admits an eigen-decomposition [8], namely,  $\mathcal{L} = U\Gamma U^T$ . Hitting time  $H_{ij}$  captures the expected number of random walk steps from node  $i$  to first hit node  $j$ . Commute time is the expected number of random walk steps from node  $i$  to first hit node  $j$ , and return to node  $i$ . Using the eigenvalues and eigenvectors of  $\mathcal{L}$ , we can compute the hitting times and commute times using the following formula [19]:

$$H_{ij} = \sum_{k>1} \frac{d}{\lambda_k} \left( \frac{\mu_{kj}^2}{d_j} - \frac{\mu_{ki}\mu_{kj}}{\sqrt{d_i d_j}} \right), \quad (5)$$

and

$$C_{ij} = \sum_{k>1} \frac{d}{\lambda_k} \left( \frac{\mu_{ki}}{\sqrt{d_i}} - \frac{\mu_{kj}}{\sqrt{d_j}} \right)^2, \quad (6)$$

where  $\mu_{kj}$  is the  $j$ th entry of the column vector  $\mu_k$ .

### 3. COMPLEX NETWORK ANALYSIS

The edges of the graph could exhibit versatile characteristics reflecting various underlying entity relations. On undirected graphs, the edges connecting nodes are symmetric, e.g., the friendship relations in Facebook, while on digraphs, the edges are in general

asymmetric, e.g., the users' following relations on Twitter, and the hyperlinking relation on the World Wide Web (WWW) network. On signed graphs, the edges carry heterogeneous weights, which could be either positive or negative, representing trust and distrust relations among OSN users. This section discusses application problems in three types of graphs, including directed graphs, undirected graphs, and signed graphs.

#### 3.1 Directed graph model

Graphs arising from many applications are directed, where entity connections in a direct graph can be categorized into two types, namely, bi-directional links (mutual connections) and unidirectional links (one-way connections), for example, the users' following relations on Twitter, and the hyperlinking relations on the World Wide Web (WWW) network. Differing from undirected graphs, the direction of links contains crucial information, which makes it challenging to model and characterize such directed complex networks. This section introduces state-of-the-art works in extending and generalizing the standard random walk theory and the intrinsically related spectral graph theory on undirected graphs to digraphs, which is further applied to solve various practical problems in wireless networks.

**Spectral graph theory for digraphs.** Given a directed graph  $G$ , its adjacency matrix is in general asymmetric, thus the relations (i.e., eq.(1) and (2)) between stationary distribution and node degrees no longer hold. Moreover, the Laplacian matrix defined in eq.(3) is generally asymmetric, thus not eigen-decomposable. [12, 14] focuses on ergodic digraph, which is strongly connected and aperiodic, i.e., there is a (directed) path from any vertex  $i$  to any other vertex  $j$ , and the Markov chain  $P$  has a unique stationary probability distribution. Then, a Diplacian (digraph laplacian)<sup>1</sup> is defined in terms of the node stationary distribution vector in stead of node degree vector,

$$\tilde{\mathcal{L}} = \Pi^{\frac{1}{2}}(I - P)\Pi^{-\frac{1}{2}}, \quad (7)$$

where  $\Pi = \text{diag}[\pi_i]$  is the diagonal stationary distribution matrix, and  $P = [p_{ij}]$  is the transition probability matrix.

Diplacian matrix  $\tilde{\mathcal{L}}$  is in general asymmetric and not eigen-decomposable. Instead, [12, 14] shows that the singular value decomposition of  $\tilde{\mathcal{L}}$ , i.e.,  $\tilde{\mathcal{L}} = U\Sigma V^T$ , captures the spectrum of a digraph, and has close connections to the hitting time and commute time representation, Green's function of  $\tilde{\mathcal{L}}$ , the normalized fundamental matrix [19]. For example, the hitting time can be computed in terms of the spectrum space formed by the singular values and vectors of  $\tilde{\mathcal{L}}$ .

$$H_{ij} = \frac{\tilde{\mathcal{L}}_{jj}^+}{\pi_j} - \frac{\tilde{\mathcal{L}}_{ij}^+}{\sqrt{\pi_i \pi_j}} = \sum_{k>1} \frac{1}{\sigma_k} \left( \frac{v_{kj} u_{kj}}{\pi_j} - \frac{v_{ki} u_{kj}}{\sqrt{\pi_i \pi_j}} \right), \quad (8)$$

where  $\sigma_i$ ,  $u_i$ , and  $v_i$  represent the  $i$ -th singular value, left and right singular vectors, respectively.

**Estimating wireless transmission costs.** In various wireless scenarios, stateless (stochastic) routing is commonly used for the ease of implementation and the adaptation to the dynamic changes of network topology. As no routing states are maintained or used, packages are forwarded in a random walk fashion in wireless networks. Given a source-destination pair  $(s, d)$ , any node in the network  $G$  may be involved in the forwarding process of a packet. Suppose that node  $i$  is the current forwarder. After node  $i$ 's transmission, a subset of its direct neighbors,  $N(i)$ , may receive the

<sup>1</sup>A unnormalized digraph Laplacian is defined in [2].

packet. For example, the next forwarder may be selected by using a random back-off mechanism where each node randomly sets a back-off timer value uniformly chosen from  $[0, t_0]$  where  $t_0$  is an appropriately chosen contention slot. Hence to track the packet traversals under stateless routing, we see that the packet stays with node  $i$  if and only if none of its neighbors receive the packet. This happens with probability  $p_{ii} = \prod_k (1 - a_{ik})$ . Otherwise, the packet transits or “walks” from node  $i$  to node  $j$ ,  $j \in N(i)$ , with probability  $p_{ij} = \frac{a_{ij}}{\sum_k a_{ik}} (1 - \prod_k (1 - a_{ik}))$ . Hence we have a Markov Chain with the transition probability matrix  $P = [p_{ij}]$  given below,

$$p_{ij} = \begin{cases} \frac{a_{ij}}{\sum_k a_{ik}} (1 - \prod_k (1 - a_{ik})) & \text{if } i \neq j \\ \prod_k (1 - a_{ik}) & \text{if } i = j. \end{cases} \quad (9)$$

It is easy to verify that  $\sum_j p_{ij} = 1$ . If the graph is strongly connected and aperiodic, the Markov chain is ergodic. Thus the hitting time  $H_{ij}$  captures exactly the expected number of hops needed to transmit a packet from node  $i$  to  $j$  with stateless routing.

To account for other transmission costs, we introduce a transition cost matrix  $T = [T_{ij}]$  associated with each one-hop transition,  $T_{ij} \geq 0, \forall i, j$ . For example, depending on the context and modeling objective,  $T_{ii}$  can be used to represent the per-node processing/transmission latency, duty cycle delay, or per-node energy consumption; where  $T_{ij}, j \in N(i)$  the one-hop forwarding latency, energy consumption, etc. Analogous to the notion of hitting time  $H_{ij}$ , we define the *hitting cost*,  $H_{ij}^s$ , (also referred as the *sojourn time* associated with  $T$ ) as the (expected) total cost (or “delay”) incurred by a random walk that starts at node  $i$  to first reach node  $j$ , where each state at any node  $k$  incurring a cost (delay)  $T_{kk}$  and each transition from node  $k$  to node  $l$  incurring a cost (delay) of  $T_{kl}$ . As in the case of  $H_{ij}$ ,  $H_{ij}^s$  satisfies the following recursive relation where  $s_i = \sum_j p_{ij} T_{ij}$  as the average transmission cost every time a packet visits.

$$H_{ij}^s = \begin{cases} \sum_{k=1}^n p_{ik} (T_{ik} + H_{kj}^s) = s_i + \sum_{k=1}^n p_{ik} H_{kj}^s & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \quad (10)$$

Hence given the appropriately defined Markov chain for a wireless routing scheme and the transition cost matrix  $H^s$ , we can use  $H_{sd}^s$  to capture the (expected) total cost of transmission when forwarding a packet from source  $s$  to destination  $d$ . We note that if  $T_{ij} = 1$  for all  $i, j$ , i.e.,  $T$  is the all-1 matrix, then  $H_{ij}^s = H_{ij}$ .

The hitting and commute costs can be computed as follows. Given an (asymmetric) transition cost matrix  $T = [T_{ij}]$ ,  $T_{ij} \geq 0$ , with  $T_{ij}$  as the per-link transition cost, and  $T_{ii}$  as the per-node transmission cost, define  $S = \text{diag}[s_i]$  a diagonal matrix with  $s_i = \sum_{j=1}^n T_{ij} p_{ij}$  as the average transmission cost every time a packet visits. We define the *normalized cost Laplacian* matrix as  $\tilde{\mathcal{L}}^s = S^{-\frac{1}{2}} \tilde{\mathcal{L}} S^{-\frac{1}{2}}$ . Let  $\tilde{\mathcal{L}}^{s+}$  be the (Penrose-Moore) pseudo-inverse of  $\tilde{\mathcal{L}}^s$ . From eq.(10), the hitting costs  $H_{ij}^s$  can be computed using  $\tilde{\mathcal{L}}^{s+}$  as stated below

$$H_{ij}^s = d^s \left( \frac{\tilde{\mathcal{L}}_{jj}^{s+}}{\pi_j s_j} - \frac{\tilde{\mathcal{L}}_{ij}^{s+}}{\sqrt{\pi_i s_i \pi_j s_j}} \right), \quad (11)$$

where  $d^s = \sum_k \pi_k s_k$ . Analogous to the commute times  $C_{ij}^s$ , we can also define the *commute costs* as  $C_{ij}^s = H_{ij}^s + H_{ji}^s$ , and they can be computed easily using eq.(11).

## 3.2 Undirected graph model

Routing is a critical operation in networks. In the context of data and sensor networks, routing strategies such as shortest-path, multi-path, and potential-based (“all-path”) routing have been developed, which intrinsically represent the tradeoff between the latency and

energy dissipation of paths used for routing, namely, shorter paths lead to better routing with low latency, while diffusing traffic along more paths generally reduces energy dissipation. Based on the connection between routing and flow optimization in a network, [17, 18] consider networks as undirected graphs and develop a unifying theoretical framework by considering flow optimization with mixed (weighted)  $L_1/L_2$ -norms as follows. Consider an  $n$ -node network as an undirected, weighted graph,  $G = (V, E, W)$ , where  $V = \{1, 2, \dots, n\}$  is the set of vertices,  $E$  is the set of edges, and each edge  $(i, j)$  is assigned a positive weight  $w_{ij}$ . As  $G$  is undirected,  $(i, j)$  and  $(j, i)$  represent the same edge in  $E$ , and  $w_{ij} = w_{ji} > 0$ . Let  $d = [s, t]$ ,  $s, t \in V$ , denote a source-destination (or source-sink) pair in the network  $G$ . A unit network flow goes from source  $s$  to destination  $d$  is mathematically defined as a function,  $X : V \times V \in R^+$ .  $\theta$  represents the trade-of parameter between  $L_1$  and  $L_2$ -norms.

### Mixed $L_1$ - and $L_2$ -norm Network Flow Optimization:

$$\min_X \sum_{i=1}^n \sum_{j=1}^n w_{ij} X_{ij}^2 + 2\theta \sum_{i=1}^n \sum_{j=1}^n w_{ij} X_{ij}.$$

s.t.  $X_{ij} \geq 0, 1 \leq i, j \leq n$ .

$$\sum_{j:(i,j) \in E} X_{ij} - \sum_{k:(k,i) \in E} X_{ki} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i = 2, \dots, n-1 \\ -1 & \text{if } i = n, \end{cases}$$

Given a trade-off parameter  $\theta \geq 0$ , denote the optimal solution as  $X^*(\theta)$ . Let  $\mathcal{R}(\theta) = (V(\theta), E(\theta))$  denote the routing graph induced by  $X^*(\theta)$ , a subgraph of  $G = (V, E)$ , where  $(i, j) \in \mathcal{R}(\theta)$  if and only if  $X_{ij}^*(\theta) > 0$ . Then,  $X^*(\theta)$  can be computed by using the graph spectrum, namely, Laplacian matrix  $L(\theta) = D(\theta) - A(\theta)$ . Moreover, by analyzing the obtained  $X^*(\theta)$  with different  $\theta$ 's, a surprising result is observed: When varying the trade-off parameter  $\theta$ , the routing graphs induced by the optimal flow solutions span from shortest-path to multi-path to all-path routing – this entire sequence of routing graphs is referred to as the routing continuum. We also develop an efficient iterative algorithm for computing the entire routing continuum. Several generalizations are also considered in [17, 18], with applications to traffic engineering and wireless sensor networks.

## 3.3 Signed Graph Model

In many online social networks, such as Slashdot and Epinions, the edges (i.e., relations between users) carry heterogeneous weights, which can be either positive or negative, representing trust (or friend) and distrust (or foe) relations. These networks are referred to as signed networks, where those signed weights generate new challenges in understanding and studying the underlying network properties. For example, the matrix representation of a signed network, namely, the adjacency matrix is no longer non-negative, and it is not clear how the information is propagated on such graphs. [9, 10] study the influence diffusion and influence maximization in social networks with both positive and negative relationships. By representing such social networks using signed directed graphs, a novel voter diffusion model is proposed, where the social influence is propagated in a random walk fashion. Initially, each node in the signed network has one of two opposite opinions, i.e., “like” and “dislike” a product. At each step, every node randomly picks one of its outgoing neighbors, and if the edge to this neighbor is positive, the node adopts the neighbor’s opinion, but if the edge is negative, the node adopts the opposite of the neighbor’s opinion. Given such a signed voter model, [9, 10] investigates if the influence propagation converges over a long term, what the convergence status is (if it

exists), and how to choose initial seed nodes to maximize the long term influence of one opinion vs the other.

Consider a signed directed graph  $G = (V, E, A)$ , where  $V$  is the set of vertices,  $E$  is the set of directed edges, and  $A$  is the signed adjacency matrix with a positive entry  $A_{ij}$  representing that  $i$  considers  $j$  as a friend or  $i$  trusts  $j$ , and a negative  $A_{ij}$  meaning that  $i$  considers  $j$  as a foe or  $i$  distrusts  $j$ . The absolute value  $|A_{ij}|$  represents the strength of this trust or distrust relationship. A signed transition matrix is defined as  $P = D^{-1}A$ , where  $D = \text{diag}[d_i]$  is the diagonal matrix and  $d_i = \sum_{j \in V} |A_{ij}|$  is the out-degree of node  $i$ . A key theorem developed in [9, 10] is the convergence analysis on power series of signed transition matrix  $P$ , namely,  $\lim_{t \rightarrow \infty} P^t$  and  $\lim_{t \rightarrow \infty} \sum_{i=0}^{t-1} P^i$ , which has intrinsic relations to the spectrum of signed graphs. Based these results, [9, 10] develop a complete characterization of the short-term and long-term dynamics of the signed voter model, and efficient algorithms to solve both short-term and long-term influence maximization problems.

#### 4. FUTURE DIRECTIONS

Spectral graph theory, as an enabling tool, can be used to solve a broad range of application problems in complex networks. Going beyond the existing works discussed above, it is interesting to investigate spectral graph theory in multi-attributed complex networks.

Entities in complex networks usually possess multiple attributes that reflect their different characteristics, for example, the numbers of users, who follow and be followed by a Twitter user, indicate the popularity and the activeness of the Twitter user, respectively. The emphasis in complex networks has been on analyzing and characterizing individual attributes separately, e.g., degree distributions as univariate distributions, which have implications on issues such as social influence, community formation, resilience, and epidemics. However, multivariate phenomena that arise as a result of repeated non-linear interactions among its participants are mostly ignored, where valuable information about the correlations across different network attributes is lost. Thus, bringing multivariate probabilistic models and analytics to bear in complex network analysis will provide a more rigorous theoretical underpinning, and will lead to deeper insights into many network phenomena. Thus, it is interesting to investigate the spectral graph properties of multi-attributed networks, with which as theoretical foundations to design efficient algorithms to characterize the multivariate statistics and structures of complex networks through sampling, e.g., joint in-and out-degree distributions. The results by analyzing the sampled data will infer correlations and causality in complex networks, which in turn have implications in revealing more precise network evolution process with respect to multiple attributes, and uncovering hidden social communities in the underlying networks.

#### 5. CONCLUSION

In this paper, we introduce the spectral graph theory, a powerful tool in analyzing various complex networks, as directed, undirected, and signed graphs. We discuss a variety of application problems in wireless networks, Internet traffic engineering, and influence diffusion in social networks. we also highlight some future directions to further extend and develop spectral graph theory in more general settings, such as multi-attributed networks. This paper aims to help the community better understand and explore this nascent area.

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#### 7. REFERENCES

- [1] D. Aldous and J. A. Fill. Reversible markov chains and random walks on graphs. <http://www.stat.berkeley.edu/~Aldous/RWG/book.html>.
- [2] D. Boley, G. Ranjan, and Z.-L. Zhang. Commute times for a directed graph using an asymmetric laplacian. *Linear Algebra and its Applications*, 435(2):224–242, 2011.
- [3] C.-K. Chau and P. Basu. Exact analysis of latency of stateless opportunistic forwarding. In *INFOCOM 2009*, pages 828–836. IEEE, 2009.
- [4] K. Chaudhuri, F. C. Graham, and A. Tsiatas. Spectral clustering of graphs with general degrees in the extended planted partition model. In *COLT*, pages 35–1, 2012.
- [5] F. R. K. Chung. Laplacians and the cheeger inequality for directed graphs. *Annals of Combinatorics*, 9(1):1–19, 2005.
- [6] F. R. K. Chung. *Spectral Graph Theory (CBMS Regional Conference Series in Mathematics, No. 92)*. Cbms Regional Conference Series in Mathematics, 2006.
- [7] L. Hogben. Spectral graph theory and the inverse eigenvalue problem of a graph. *Electronic Journal of Linear Algebra*, 14(1):3, 2005.
- [8] R. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1st edition, 1985.
- [9] Y. Li, W. Chen, Y. Wang, and Z.-L. Zhang. Influence diffusion dynamics and influence maximization in social networks with friend and foe relationships. In *WSDM*, pages 1–10. ACM, 2013.
- [10] Y. Li, W. Chen, Y. Wang, and Z.-L. Zhang. Voter model on signed social networks. *Internet Mathematics*, PP(99):1–50, 2014.
- [11] Y. Li, W. Chen, and Z.-L. Zhang. Optimal forwarder list selection in opportunistic routing. In *MeshTech*, pages 670–675. IEEE, 2009.
- [12] Y. Li and Z.-L. Zhang. Random walks on digraphs: A theoretical framework for estimating transmission costs in wireless routing. In *INFOCOM*, pages 1–9. IEEE, 2010.
- [13] Y. Li and Z.-L. Zhang. Random walks on digraphs, the generalized digraph laplacian and the degree of asymmetry. *WAW*, pages 74–85, 2010.
- [14] Y. Li and Z.-L. Zhang. Digraph laplacian and the degree of asymmetry. *Internet Mathematics*, 8(4), 2012.
- [15] Y. Li and Z.-L. Zhang. Random walks and green’s function on digraphs: A framework for estimating wireless transmission costs. *IEEE/ACM Transactions on Networking*, PP(99):1–14, 2012.
- [16] Y. Li, Z.-L. Zhang, and J. Bao. Mutual or unrequited love: Identifying stable clusters in social networks with uni- and bi-directional links. In *WAW*, pages 113–125, 2012.
- [17] Y. Li, Z.-L. Zhang, and D. Boley. The routing continuum from shortest-path to all-path: A unifying theory. In *ICDCS*, pages 847–856. IEEE, 2011.
- [18] Y. Li, Z.-L. Zhang, and D. Boley. From shortest-path to all-path: The routing continuum theory and its applications. *IEEE Transactions on Parallel and Distributed Systems*, 25(7):1745–1755, 2014.
- [19] L. Lovász. Random walks on graphs: A survey. *Combinatorics*, 2:1–46, 1993.
- [20] D. Zhou, J. Huang, and B. Schölkopf. Learning from labeled and unlabeled data on a directed graph. In *ICML*, pages 1036–1043, 2005.