A Game Theoretic Model for the Formation of Navigable Small-World Networks*

Zhi Yang[§] and Wei Chen[†]
[§]Computer Science Department, Peking University, Beijing, China
[†]Microsoft Research, Beijing, China
yangzhi@net.pku.edu.cn, weic@microsoft.com

ABSTRACT

Kleinberg proposed a family of small-world networks to explain the navigability of large-scale real-world social networks. However, the underlying mechanism that drives real networks to be navigable is not yet well understood. In this paper, we present a game theoretic model for the formation of navigable small world networks. We model the network formation as a game in which people seek for both high reciprocity and long-distance relationships. We show that the navigable small-world network is a Nash Equilibrium of the game. Moreover, we prove that the navigable small-world equilibrium tolerates collusions of any size and arbitrary deviations of a large random set of nodes, while non-navigable equilibria do not tolerate small group collusions or random perturbations. Our empirical evaluation further demonstrates that the system always converges to the navigable network even when limited or no information about other players' strategies is available. Our theoretical and empirical analyses provide important new insight on the connection between distance, reciprocity and navigability in social networks.

Categories and Subject Descriptors

G.2.2 [**Discrete Mathematics**]: Graph Theory—*Network problems*

General Terms

Theory

Keywords

small-world network, game theory, navigability, reciprocity

1. INTRODUCTION

In 1967, Milgram published his work on the now famous small-world experiment [30]: he asked test subjects to forward a letter to their friends in order for the letter to reach a person not known to the initiator of the letter. He found that on average it took only six hops to connect two people in U.S., and coined the term *six-degree of separation.* This seminal work inspired numerous researches on the small-world phenomenon and small-world models, which last till the present day of information age.

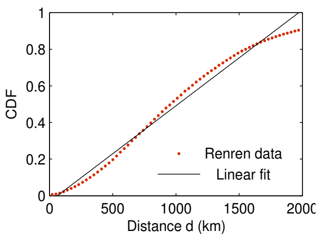
In [36] Watts and Strogatz investigated a number of real-world networks such as film actor networks and power grids, and showed that many networks have both low diameter and high clustering (meaning two neighbors of a node are likely to be neighbors of each other), which is different from randomly wired networks. They thus proposed a small-world model in which nodes are first placed on a ring or a grid with local connections, and then some connections are randomly rewired to connect to long-range contacts in the network. The local and long-range connections can also be viewed as strong ties and weak ties respectively in social relationships originally proposed by Granovetter [14, 13].

Kleinberg notices an important discrepancy between the small-world model of Watts and Strogatz and the original Milgram experiment: the latter shows not only that the average distance between nodes in the network are small, but also that nodes can efficiently navigate in the network with only local information. To address this issue, Kleinberg adjusted Watts-Strogatz model so that the long-range connections are selected not uniformly at random among all nodes but inversely proportional to a power of the grid distance between the two end points of the connection [21].

More specifically, Kleinberg modeled a social network as composed of n^k nodes on a k-dimensional grid, with each node having local contacts to other nodes in its immediate geographic neighborhood. Each node u also establishes a number of long-range contacts, and a long-range link from u to v is established with probability proportional to $d_M(u,v)^{-r}$, where $d_M(u,v)$ is the grid distance between u and v, and $r \geq 0$ is the model parameter indicating how likely nodes prefer to connect to remote nodes, which we call *connection preference* in the paper. Watts-Strogatz model corresponds to the case of $r = 0$, and as r increases, nodes are more likely to connect to other nodes in their vicinity. Kleinberg modeled Milgram's experiment as decentralized greedy routing in such networks, in which each node only forwards messages to one of its neighbors with coordinate closest to the target node. He showed that when $r = k$, greedy routing can be done efficiently in $O(\log^2 n)$ time in expectation, but for any $r \neq k$, it requires $O(n^c)$ time for some constant c depending on r. Therefore, the small world at the critical value of $r = k$ is meant to model the real network, and we call it the *navigable small-world network.*

*The work is done when the first author is a visiting researcher at Microsoft Research Asia.

WWW 2015, May 18–22, 2015, Florence, Italy.
ACM 978-1-4503-3469-3/15/05.
http://dx.doi.org/10.1145/2736277.2741110.



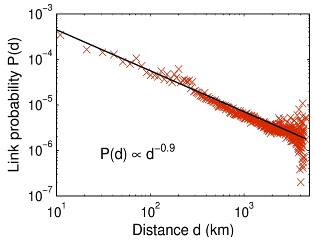


Figure 1: The fraction of nodes within distance d in Renren.

Figure 2: Friendship probability vs. distance in Renren.

After Kleinberg's theoretical analysis, a number of empirical studies have been conducted to verify if real networks indeed have connection preference close to the critical value that allows efficient greedy routing [28, 1, 7, 10, 34]. Since real network population is not evenly distributed geographically as in the Kleinberg's model, Liben-Nowell et al. [28] proposed to use the *fractional dimension* α, defined as the best value to fit $|\{w : d_M(u,w) \leq d_M(u,v)\}| = c \cdot d_M(u,v)^\alpha$. They showed that when the connection preference $r = \alpha$, the network is navigable. They then studied a network of 495,836 LiveJournal users in the continental United States who list their hometowns, and find that $\alpha \approx 0.8$ while $r = 1.2$, reasonably close to α. We apply the same approach to a ten million node Renren network [19, 38], one of the largest online social networks in China. We map the hometown listed in users' profiles to (longitude, latitude) coordinates. The resolution of our geographic data is limited to the level of towns and cities and thus we cannot get the exact distance of nodes within 10km. We found that $\alpha \approx 1$ (Figure 1) and $r \approx 0.9$ (Figure 2) in Renren network. Other studies [1, 7, 10, 34] also reported connection preference r to be close to 1 in other online social networks (including Gowalla, Brightkite and Facebook). Even though they did not report the fractional dimension, from both the LiveJournal data in [28] and our Renren data, it is reasonable to believe that the fractional dimension is also close to 1. Therefore, empirical evidences all suggest that the real-world social networks indeed have connection preference close to the critical value and the network is navigable.

A natural question to ask next is how navigable networks naturally emerge? What are the forces that make the connection preference become close to the critical value? As Kleinberg pointed out in his survey paper [22] when talking about the above striking coincidence between theoretical prediction and empirical observation, "it suggests that there may be deeper phenomena yet to be discovered here". There are several studies trying to explain the emergence of navigable small-world networks [29, 16, 8, 33, 5], mostly by modeling certain underlying node or link dynamics (see related work for more details).

In this paper, we tackle the problem in a novel way using game-theoretic approach, which is reasonable in modeling individual behaviors in social networks without central coordination. One key insight we have is that connection preference r is not a global preference but individual's own preference — some prefer to connect to more faraway nodes while others prefer to connect to nearby nodes. Therefore, we establish *small-world formation games* where individual node u's strategy is its own connection preference r_u (Section 3). This game formulation is different from most existing network formation games where individuals' strategies are creating actual links in the network (c.f. [35]). It allows us to directly explore the entire parameter space of connection preferences and answer the question on why nodes end up choosing a particular parameter setting leading to the navigable small world.

In terms of payoff functions, we first consider minimizing greedy routing distance to other nodes as the payoff, since it directly corresponds to the goal of navigable networks. However, Gulyás et al. [15] prove that with this payoff the navigable networks cannot emerge as a equilibrium for the one-dimensional case. Our empirical analysis also indicates that nodes will converge to random networks ($r_u = 0, \forall u$) rather than navigable networks for higher dimensions. Our empirical analysis further shows that if we adjust the payoff with a cost proportional to the grid distance of remote connections, the equilibria are sensitive to the cost factor.

The above unsuccessful attempt suggests that besides the goal of shortening distance to remote nodes, some other natural objective may be in play. Reciprocity is regarded as a basis mechanism that creates stable social relationships in a person's life [12]. A number of prior works [18, 28, 31] also suggest that people seek reciprocal relationships in online social networks. Therefore, we propose a payoff function that is the product of average distance of nodes to their long-range contacts and the probability of forming reciprocal relationship with long-range contacts. We call this game distance-reciprocity balanced (DRB) game. In practice, increasing relationship distance captures that individuals attempt to create social bridges by linking to "distant people", which can help them search for and obtain new resources. Meanwhile, increasing reciprocity captures that individuals look at social bonds by linking to "people like them", which could help them preserve or maintain resources. Therefore, the DRB game is natural since it captures sources of bridging and bonding social capital in building social integration and solidarity [9].

Even though the payoff function for the DRB game is very simple, our analysis demonstrates that it is extremely effective in producing navigable small-world networks as the equilibrium structure. In theoretical analysis (Section 4), we first show that both navigable small world ($r_u = k, \forall u$) and random small world ($r_u = 0, \forall u$) are the only two uniform Nash equilibria of the DRB game. Although we do not know whether non-uniform equilibria exist, we prove that the navigable small world is a strong Nash equilibrium, which means that it tolerates collusion of any size trying to gain better payoff, and it also tolerates arbitrary deviations (without the objective of increasing anyone's payoff) of large groups of random deviators. In contrast, any non-navigable equilibrium such as the random small world or possible non-uniform equilibria does not tolerate either collusions or random perturbations of a small group of nodes. Our theoretical analysis provides strong support that navigable small-world network is the unique stable equilibrium that would naturally emerge in the DRB game.

We further conduct empirical evaluations to cover more realistic game scenarios not covered by our theoretical analysis (Section 5). We first test random perturbation cases and show that arbitrary initial profiles always converge to the navigable equilibrium in a few steps, while a very small random perturbation (less than theoretical prediction) of the random small world causes it to quickly converge back to the navigable equilibrium. Next, we simulate more realistic scenarios where nodes have limited or no information about other nodes' strategies. We show that if they only learn their friends' strategies (with some noise), the system still converges close to the navigable equilibrium in a small number of steps. Finally, even when the node has no information about other players' strategies and can only use its obtained payoff as feedback to search for the best strategy, the system still moves close to the navigable equilibrium within a few hundred steps (in the 100×100 grid). These empirical results further demonstrate the robustness of the navigable small world in the DRB game.

In summary, our contributions are the following: (a) we propose the small-world formation game and design a balanced distance-reciprocity payoff function to explain the navigability of real social networks; (b) we conduct comprehensive theoretical and empirical analysis to demonstrate that navigable small world is a unique robust equilibrium that would naturally emerge from the game under both random perturbation and strategic collusions; and (c) our game reveals a new insight between distance, reciprocity and navigability in social networks, which may help future research in uncovering deeper phenomena in navigable social networks. To our best knowledge, this is the first game theoretic study on the emergence of navigable small-world networks, and the first study that linking relationship reciprocity with network navigability.

Additional related work. We provide additional details of prior works on explaining the emergence of navigable small-world networks, and other related studies not covered in the introduction.

Some studies try to explain navigability by assuming that nodes form links to optimize for a particular property. Mathias et al. [29] assume that users try to make trade-off between wiring and connectivity. Hu et al. [16] assume that people try to maximize the entropy under a constraint on the total distances of their long-range contacts. These works rely on simulations to study the network dynamics. Moreover, the navigability of a network is sensitive to the weight of wiring cost or the distance constraint, and it is unlikely that navigable networks as defined by Kleinberg [21] would naturally emerge.

Another type of works propose node/link dynamics that converge to navigable small-world networks. Clauset and Moore [8] propose a rewiring dynamic modeling a Web surfer such that if the surfer does not find what she wants in a few steps of greedy search, she would rewire her long-range contact to the current end node of the greedy search. They use simulations to demonstrate that a network close to Kleinberg's navigable small world will emerge after long enough rewiring rounds. Sandberg and Clarke [33] propose another rewiring dynamic where with an independent probability of p each node on a greedy search path would rewire their long-range contacts to the search target, and provide a partial analysis and simulations showing that the dynamic converges to a network close to the navigable small world. Chaintreau et al. [5] use a move-and-forget mobility model, in which a token starting from each node conducts a random walk (move) and may also go back to the starting point (forget), and use the distribution of the token on the grid as the distribution of the long-range contacts of the starting node. They provide theoretical analysis showing that the move-and-forget model with a particular harmonic forget function converges close to the navigable small world. However, it is unclear if the harmonic forget function used is natural in practice and what is the effect of other forget functions.

The approach taken by these studies can be viewed as orthogonal and complementary to our approach: they aim at using natural dynamics (rewiring or mobility dynamics) to explain navigable small world, while we focus on directly exploring the entire parameter space of connection preferences of nodes and use game theoretic approach to show, both theoretically and empirically, that the nodes would naturally choose their connection preferences to form the navigable small world. Moreover, all the prior studies only show that they converge approximately to the navigable small world, while in our game the navigable small world is precisely the only robust equilibrium. Finally, none of these works introduce reciprocity in their model and we are the first to link reciprocity with navigability of the small world.

Some studies use hyperbolic metric spaces or graphs to try to explain navigability in small-world networks (e.g. [4, 32, 23, 24, 6]). However, they do not explain why connection preferences in real networks are around the critical value and how navigable networks naturally emerge. In particular, Chen et al. [6] show that the navigable small world in Kleinberg's model does not have good hyperbolicity.

A number of network dynamics are proposed to address general network evolution, but they do not address network navigability in particular. For example, models in [3, 27, 26, 11] leverage preferential attachment or triangle closure mechanisms to capture power-law degree or high clustering coefficient, and other models [2, 20] capture spatial effects using a gravity model, balancing the effect of spatial distance with other node properties (e.g., node degree).

2. PRELIMINARIES

In this section, we present the Kleinberg's small-world model and some basic concepts of a noncooperative game.

2.1 Kleinberg's Small-World Model

Let $V = \{(i, j) : i, j \in [n] = \{1, 2, \ldots, n\}\}$ by the set of n^2 nodes forming an $n \times n$ grid. For convenience, we consider the grid with wrap-around edges connecting the nodes on the two opposite sides, making it a torus. For any two nodes $u = (i_u, j_u)$ and $v = (i_v, j_v)$ on this wrap-around grid, the *grid distance* or *Manhattan distance* between u and v is defined as $d_M(u, v) = min\{|i_v - i_u|, n - |i_v - i_u|\} + min\{|j_v - j_u|, n - |j_v - j_u|\}$.

The model has two universal constants $p, q \geq 1$, such that (a) each node has undirected edges connecting to all other nodes within lattice distance p, called its *local contacts*, and (b) each node has q random directed edges connecting to possibly faraway nodes in the grid called its *long-range contacts*, drawn from the following distribution. Each node u has a *connection preference* parameter $r_u \geq 0$, such that the i-th long-range edge from u has endpoint v with probability proportional to $1/d_M(u, v)^{r_u}$, that is, with probability $p_u = d_M(u, v)^{-r_u}/c(r_u)$, where $c(r_u) = \sum_{\forall v \neq u} d(u, v)^{-r_u}$ is the normalization constant. Let $\mathbf{r}$ be the vector of r_u values on all nodes. We use $\mathbf{r} \equiv s$ to denote $r_u = s, \forall u \in V$.

Greedy routing on the small-world network from a source node u to a target node v is a decentralized algorithm starting at node u, and at each step if routing reaches a node w, then w selects one node from its local and long-range contacts that is closest to v in grid distance as the next step in the routing path, until it reaches v. In [21], Kleinberg shows that when $\mathbf{r} \equiv 2$, the expected number of greedy routing steps (called *delivery time*) is $O(\log^2 n)$, but when $\mathbf{r} \equiv s \neq 2$, it is $\Omega(n^c)$ for some constant c related to s.

The above model can be easily extended to k dimensional grid (with wraparound) for any $k = 1, 2, 3, \ldots$, where each long range contact is still established with probability proportional to $1/d_M(u, v)^{r_u}$. It is shown that $\mathbf{r} \equiv k$ is the critical value allowing efficient greedy routing. Henceforth, we call Kleinberg's small world with $\mathbf{r} \equiv k$ the *navigable small world*. Another special network is $\mathbf{r} \equiv 0$, in which every node's long-range contacts are selected among all nodes uniformly at random, and we refer it as the *random small world*. We use $K(n, k, p, q, \mathbf{r})$ to refer to the class of Kleinberg random graphs with parameters n, k, p, q, and $\mathbf{r}$.

2.2 Game and Solution Concepts

A game is described by a system of players, strategies and payoffs. We denote a game by $\Gamma = (S_u, \pi_u)_{u \in V}$, where V represents a finite set of players, S_u is the set of strategies of player u, and $\pi_u : \mathcal{S} \to \mathbb{R}$ is the payoff function of node u, with $\mathcal{S} = S_1 \times S_2 \times \ldots \times S_n$. An element $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathcal{S}$ is called a *strategy profile*.

Let $\mathcal{C} = 2^V \setminus \emptyset$ denote the set of all coalitions. For each coalition $C \in \mathcal{C}$, let $-C = V \setminus C$, and if $C = \{u\}$, we denote $-C$ by $-u$. We also denote by $\mathcal{S}_C$ the set of strategies of players in coalition C, and $\mathbf{s}_C$ the partial strategy profile of $\mathbf{s}$ for nodes in C.

Definition 1 (Best response). *Player u's strategy $s_u^* \in S_u$ is a best response to the strategy profile $\mathbf{s_{-u}} \in \mathcal{S}_{-u}$ if*

$$\pi_u(s_u^*, \mathbf{s_{-u}}) \geq \pi_u(s_u, \mathbf{s_{-u}}), \forall s_u \in S_u \setminus \{s_u^*\},$$

Moreover, if "$\geq$" above is actually "$>$" for all $s_u \neq s_u^*$, then s_u^* is the unique best response to $\mathbf{s_{-u}}$.

Nash equilibrium (NE) for a strategic game is a strategy profile such that each player's strategy is a best response to the other players' strategies.

Definition 2 (Nash equilibrium). *A strategy profile $\mathbf{s}^* \in \mathcal{S}$ is a Nash equilibrium if for every player $u \in V$, s_u^* is a best response to $\mathbf{s}_{-u}^*$; $\mathbf{s}^*$ is a strict Nash equilibrium if for every player $u \in V$, s_u^* is the unique best response to $\mathbf{s}_{-u}^*$.*

While in an NE no player can improve its payoff by unilateral deviation, some of the players may benefit (sometimes substantially) from forming alliances/coalitions with other players. The *strong Nash equilibrium (SNE)* is a strategy profile for which no coalition of players has a joint deviation that improves the payoff of each member of the coalition.

Definition 3 (Strong Nash equilibrium). *For a real number $f \in (0, 1]$, a strategy profile $\mathbf{s}^* \in \mathcal{S}$ is an f-strong Nash equilibrium if for all $C \in \mathcal{C}$ with $|C| \leq f|V|$, there does not exist any $s_C \in \mathcal{S}_C$ such that*

$$\forall u \in C, \pi_u(\mathbf{s}_C, \mathbf{s}_{-C}^*) \geq \pi_u(\mathbf{s}^*), \exists u \in C, \pi_u(\mathbf{s}_C, \mathbf{s}_{-C}^*) > \pi_u(\mathbf{s}^*).$$

When $f = 1$, we simply call $\mathbf{s}^*$ the strong Nash equilibrium.

When restricting $|C| = 1$, SNE falls back to NE; when restricting $C = V$, SNE means *Pareto-optimal*, which requires that no player can improve her payoff without decreasing the payoff of someone else. Therefore, SNE is a very strong equilibrium concept allowing collusions of any size.

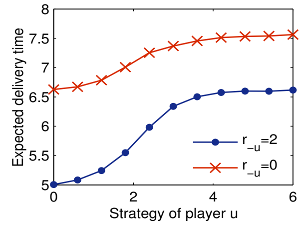


Figure 3: The expected delivery time for a player u with different strategy.

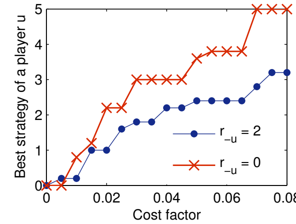


Figure 4: The best response of a player u given different cost factor.

3. SMALL-WORLD FORMATION GAMES

Connection preference r_u in Kleinberg's model reflects u's intention in establishing long-range contacts: When $r_u = 0$, u chooses its long-range contacts uniformly among all nodes in the grid; as r_u increases, the long-range contacts of u become increasingly clustered in its vicinity on the grid. Our insight is to treat connection preference as node's strategy in a game setting and study the game behavior.

More specifically, we model this via a non-cooperative game among nodes in the network. First, we assume that each r_u is taken from a discrete set $\Sigma = \{0, \gamma, 2\gamma, 3\gamma, \ldots, \}$, where γ represents the granularity of connection preference and is in the form of $1/g$ for some positive integer $g \geq 2$. Using discrete strategy set avoids nuances in continuous strategy space and is also reasonable in practice since people are unlikely to make infinitesimal changes. Next, we model the small-world network formation as a game $\Gamma = (\Sigma, \pi_u)_{u \in V}$, where V is the set of nodes in the grid, connection preference $r_u \in \Sigma$ is the strategy of a player u, and π_u is the payoff function of u. Our objective is to study intuitively appealing payoff functions π_u and find one that would allow the navigable small-world network to emerge.

3.1 Routing-based Payoff

As navigable small world achieves best greedy routing efficiency, it is natural to consider the payoff function as the expected delivery time to the target in greedy routing. Given the strategy profile $\mathbf{r} \in \mathcal{S}$, let $t_{uv}(r_u, \mathbf{r}_{-u})$ be the expected delivery time from source u to target v via greedy routing. The payoff function is given by:

$$\pi_u(r_u, \mathbf{r_{-u}}) = -\sum_{\forall v \neq u} t_{uv}(r_u, \mathbf{r}_{-u}). \quad (1)$$

We take a negation on the sum of expected delivery time because nodes prefer shorter delivery time.

Although the above payoff function is intuitive and simple, it has some serious issues. Prior work [15] has already proved that, with the length of greedy paths as the payoff, player u's best response is to link uniformly (i.e., $r_u = 0$) for the one-dimensional case. For high dimensions, Figure 3 shows the expected delivery time for a single node u at a 100×100 grids, where each node generates $q = 10$ links. We see that when other nodes fixed their strategy (e.g., $\mathbf{r}_{-u} \equiv 2$), the best strategy of a single node u is 0. More tests on different initial conditions reach the same result that the system will converge to the random small-world networks. The intuitive reason is that to reach other nodes quickly, it is better for a node to evenly spread its long-range contacts in the network.

This is inconsistent with empirical evidence that real-world networks are navigable ones. ([28, 7, 1, 10, 25, 17]).

In practice, creating and maintaining long-range links have higher costs, so one may adapt the above payoff function by adding the grid distances of long-range contacts as a cost term in the payoff function:

$$\pi_u(r_u, \mathbf{r_{-u}}) = -\sum_{\forall v \neq u} t_{uv}(r_u, \mathbf{r}_{-u}) - \lambda \sum_{v \neq u} p_u(v, r_u) d_M(u, v), \quad (2)$$

where λ is a factor controlling the long range cost. A larger λ means users are more concerned with distance costs. Figure 4 shows that the best strategy of a user u is significantly influenced by the cost factor. Similar result is also shown in [15]. Thus, it is unclear if the navigable small-world network can naturally emerge from this type of games.

3.2 Distance-Reciprocity Balanced Payoff

The previous section demonstrates that seeking short routing distance alone cannot explain the emergence of navigable small world, and thus people in the social network must have some other objective to achieve. Reciprocity is regarded as a basis mechanism that creates stable social relationships in the real world [12]. Several empirical studies [18, 28, 31] also show that high reciprocity is also a typical feature present in real small-world networks (such as Flickr, YouTube, LiveJournal, Orkut and Twitter).

Therefore, we consider the payoff of a user u as the following balanced objective between distance and reciprocity:

$$\pi_u(r_u, \mathbf{r_{-u}}) = \left(\sum_{\forall v \neq u} p_u(v, r_u) d_M(u, v) \right) \times \left(\sum_{\forall v \neq u} p_u(v, r_u) p_v(u, r_v) \right), \quad (3)$$

where $\sum_{\forall v \neq u} p_u(v, r_u) d_M(u, v)$ is the mean grid distance of u's long-range contacts, and $\sum_{\forall v \neq u} p_u(v, r_u) p_v(u, r_v)$ is the mean probability for u to form bi-directional links with its long-range contacts, i.e., reciprocity. We refer the small-world formation game with payoff function in Eq.(3) the *Distance-Reciprocity Balanced (DRB)* game.

The payoff function in Eq.(3) reflects two natural objectives users in a social network want to achieve: first, they want to connect to remote nodes, which may give them diverse information as in the famous "the strength of weak tie argument" by Granovetter [14]; second, they want to establish reciprocal relationship which are more stable in long term. However, these two objectives can be in conflict for a node u when others prefer linking in their vicinity (i.e., other nodes v choosing positive exponent r_v). In this case, faraway long-range contacts are less likely to create reciprocal links. Therefore, node u should obtain the maximum payoff when it achieves a balance between the two objectives. We use the simple product of distance and reciprocity objectives to model this balancing behavior.

4. PROPERTIES OF THE DRB GAME

In this section, we conduct theoretical analysis to discover the properties of the DRB game. When player u's strategy r_u is the unique best response to a strategy profile $\mathbf{r}_{-u}$, we denote this unique best response as $B_u(\mathbf{r}_{-u})$.

4.1 Equilibrium Existence in DRB Game

We first show that navigable small-world network is a Nash Equlibrium of the DRB game.

THEOREM 1. *For the DRB game in a k-dimensional grid, the following is true for sufficiently large n:* [1] *For every node $u \in V$, every strategy profile $\mathbf{r}$, and every $s \in \Sigma$, if $\mathbf{r}_{-u} \equiv s$, then u has a unique best response to $\mathbf{r}_{-u} \equiv s$:*

$$B_u(\mathbf{r}_{-u} \equiv s) = \begin{cases} k & \text{if } s > 0, \\ 0 & \text{if } s = 0. \end{cases}$$

PROOF (SKETCH). The intuition is as follows. When $s > 0$, all other nodes prefer long-range contacts in their vicinity. In this case, when node u chooses $0 \leq r_u < k$, it achieves good average grid distance to long-range contacts but its long-range contact v is unlikely to have the reciprocal link to u because u is not likely to be in the vicinity of v. On the other hand, if u chooses $r_u > k$, it achieves good reciprocity but its average grid distance to long-range contacts is low. The case of $r_u = k$ provides the best balance between grid distance to long-range contacts and reciprocity. The detailed proof is included in Appendix A.

When $s = 0$, all others nodes link uniformly. In this case, the reciprocity for node u becomes a constant independent of its strategy r_u. Thus, r_u should be selected to maximize average distance of u's long-range contacts, which leads to $r_u = 0$. The detailed proof for the case of $s = 0$ is included in [37]. □

Theorem 1 shows that when all other nodes use the same nonzero strategy s, it is strictly better for u to use strategy k; when all other nodes uniformly use the 0 strategy, it is strictly better for u to also use 0 strategy. When setting $s = k$ and $s = 0$, we have:

COROLLARY 2. *For the DRB game in the k-dimensional grid, the navigable small-world network ($\mathbf{r} \equiv k$) and the random small-world network ($\mathbf{r} \equiv 0$) are the two strict Nash equilibria for sufficiently large n, and there are no other uniform Nash equilibria.*

The above analysis shows that DRB game has two uniform Nash equilibria $\mathbf{r} \equiv k$ and $\mathbf{r} \equiv 0$, corresponding to navigable and random small-world networks, respectively. Other non-uniform equilibria may exist. Given that multiple equilibria exist, we need to further investigate if the navigable small world possess further properties making it the likely choice in practice. This is the task of the remaining sections.

4.2 Equilibrium Stability in DRB Game

In this section, we show the important results that the navigable small-world network is stable in terms of tolerating both collusions of any group of players and arbitrary deviations of random players' strategies, while other non-navigable equilibria do not tolerate either collusions or random perturbations of a small group of players.

We first show that the navigable small-world network tolerates collusion of players of any size.

[1]Technically, a statement being true for sufficiently large n means that there exists a constant $n_0 \in \mathbb{N}$ that may only depend on model constants such as k and γ, such that for all $n \geq n_0$ the statement is true in the grid with parameter n.

THEOREM 3. *For the DRB game in the k-dimensional grid, the navigable small-world network ($\mathbf{r} \equiv k$) is a strong Nash equilibrium for sufficiently large n.*

PROOF (SKETCH). We prove a slightly stronger result — any node u in any strategy profile $\mathbf{r}$ with $r_u \neq k$ has strictly worse payoff than its payoff in the navigable small world. Intuitively, when u deviates to $0 \leq r_u < k$, its loss on reciprocity would outweigh its gain on link distance; when u deviates to $r_u > k$, its loss on link distance is too much to compensate any possible gain on reciprocity. The detailed proof is in Appendix B. □

The above theorem shows that the navigable small-world equilibrium is not only immune to unilateral deviations, but also to deviations by coalitions, and in particular it is Pareto-optimal, such that no player can improve her payoff without decreasing the payoff of someone else.

Next, we would like to see if the navigable equilibrium can also tolerate deviations of a random set of players, even if the deviations could be arbitrary and there is no guarantee that deviated players are better off. To do so, we define the following δ-deviation Nash equilibrium, which intuitively means that even if each player has an independent probability of $p \leq \delta$ to deviate to an arbitrary strategy, the unique best response of every node u after the deviation is still $\mathbf{r}_u^*$.

DEFINITION 4 (δ-DEVIATION NASH EQUILIBRIUM). *Let $D_p \subseteq V$ be a random set of nodes where each node $u \in V$ is independently selected to be in D_p with probability p. An NE $\mathbf{r}^* \in \mathcal{S}$ is a δ-deviation NE if for any $0 < p \leq \delta$, with probability at least $1 - 1/n$, for all $u \in V$, all $r_u \in \Sigma \setminus \{r_u^*\}$, all $\mathbf{r}'_{\mathbf{D_p}} \in \Sigma^{|D_p|}$,*

$$\pi_u(r_u^*, \mathbf{r}^*_{V\setminus(D_p\cup\{u\})}, \mathbf{r}'_{\mathbf{D_p}}) > \pi_u(r_u, \mathbf{r}^*_{V\setminus(D_p\cup\{u\})}, \mathbf{r}'_{\mathbf{D_p}}),$$

where the probability is taken from the probability space of D_p.

The equilibrium is more robust when δ is larger. With the above definition, we have the following theorem:

THEOREM 4. *For the DRB game in a k-dimensional grid ($k > 1$), For any constant ε with $0 < \varepsilon < \gamma/4$, there exists $n_0 \in \mathbb{N}$ (depending only on k, γ, and ε), for all $n \geq n_0$, the navigable small-world network ($\mathbf{r} \equiv k$) is a δ-deviation NE for $\delta = 1 - n^{-\varepsilon}$.*

PROOF (SKETCH). The independently selected deviation node set D_p satisfies that with high probability, for any node u, at sufficiently many distance levels from u there are enough fraction of non-deviating nodes. We then show that u obtains higher order payoff just from these non-deviating nodes than any possible payoff she could get from any possible deviation. The full proof is in [37]. □

Notice that δ is close to 1 when n is sufficiently large, meaning that the navigable equilibrium tolerates arbitrary deviations from a large number of random nodes.

After showing that the navigable small-world is very stable, we now examine the stability of other possible equilibria.

THEOREM 5. *For the DRB game in a k-dimensional grid ($k > 1$), For any constant ε with $0 < \varepsilon < \gamma/4$, there exists $n_0 \in \mathbb{N}$ (depending only on k, γ, and ε), for all $n \geq n_0$, any possible NE $\mathbf{r} \not\equiv k$ is not an f-strong NE for $f = 2/n^{\varepsilon}$. Moreover, we can find a class of coalition sets C such that every node $u \in C$ would deviate to $r_u = k$, and then the best response of all nodes in the next step is to select k.*

PROOF (SKETCH). The proof is based on Theorem 4. We notice that a large random set D of nodes deviating arbitrarily from $\mathbf{r} \equiv k$ can be viewed as a small random set $V \setminus D$ of nodes collude together to deviate to $\mathbf{r}_D \equiv k$ from an arbitrary strategy profile. We then bound the size of coalition $|V \setminus D|$ using Chernoff bound. The full proof is in [37]. □

When n is sufficiently large, we see that f is close to 0, which means that the collusion by a small portion of players could drive the system out of the current non-navigable equilibrium and make the system quickly converge to the navigable equilibrium. Therefore, non-navigable equilibria are not stable under collusions of small groups of nodes.

In terms of random perturbation, a symmetric view of Theorem 4 already implies that if a small number of random nodes could perturb to strategy k from an arbitrary strategy profile, then the best response of all nodes after the perturbation is to select k. In addition, we show below that for the random small-world network, which is shown to be another NE, it does not tolerate a small set of random nodes randomly perturbing to a finite set of other strategies (not necessarily including k). Again, in this case the unique best response for every node after the perturbation is to select k.

THEOREM 6. *For the DRB game in a k-dimensional grid ($k > 1$) with the initial strategy profile $\mathbf{r} \equiv 0$ and a finite perturbed strategy set $S \subset \Sigma$ with at least one non-zero entry ($0 < \max S \leq \beta$), for any constant ε with $0 < \varepsilon < \gamma$, there exists $n_0 \in \mathbb{N}$ (depending only on k, γ, and ε), for all $n \geq n_0$, if for any $u \in V$, any $s \in S \setminus \{0\}$, r_u is perturbed to s with independent probability of $p \geq 1/n^{\frac{(k-1)\varepsilon}{k+\beta}}$, then with probability $1 - 1/n$,*

$$B_u(\mathbf{r'}_{-u}) = k, \forall u \in V,$$

where $\mathbf{r}'$ is the strategy profile after the perturbation.

PROOF (SKETCH). We consider the gain of a node u when selecting $r_u = k$ separately from each group of nodes with the same strategy after the perturbation, and then apply the results in Theorem 1. See [37] for details. □

An example of the above perturbation assumption is that each node is selected independently with probability $|S|/n^{\frac{(k-1)\varepsilon}{k+\beta}}$ for perturbation, and perturbed strategy is selected from S uniformly at random. Note that $|S|/n^{\frac{(k-1)\varepsilon}{k+\beta}}$ is very small for large enough n and a finite perturbed strategy set S, which implies that the best response of any node u in the perturbed profile becomes $r_u = k$ as long as a small number of random nodes are perturbed to a finite set of nonzero strategies. Suppose that every node chooses his/her best response against others simultaneously, the random small-world NE ($\mathbf{r} \equiv 0$) would be switched to the navigable small-world NE ($\mathbf{r} \equiv k$) in just one step after the perturbation.

4.3 Implications from Theoretical Analysis

Combining the above theorems together, we obtain a better understanding of how the navigable small-world network is formed. From any arbitrary initial state, best response dynamic drives the system toward some equilibrium, with the navigable small world as one of them (Corollary 2). Even if the systems temporarily converges to a non-navigable equilibrium, the state will not be stable — either a small-size

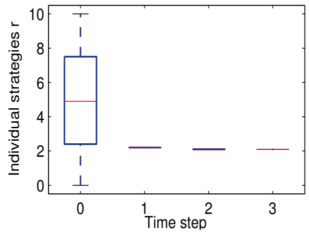


Figure 5: The return to navigable small-world NE (perturbed probability p=1).

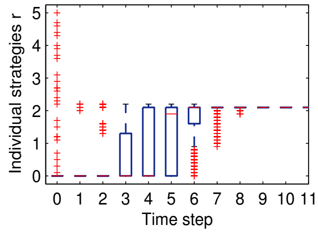


Figure 6: From random NE to small-world NE (perturbed probability p=0.01).

collusion (Theorem 5) or a small-size random perturbation (Theorem 6) would make the system leave the current equilibrium and quickly enter the navigable equilibrium. Once entering the navigable equilibrium, it is very hard for the system to move away from it — no collusion of any size would drive the system away from this equilibrium (Theorem 3), and even if a large random portion of nodes deviate arbitrarily the system still converge back to the navigable equilibrium in the next step (Theorem 4). These theoretical results strongly support that the navigable small world is the unique stable system state, which suggests that the fundamental balance between reaching out to remote people and seeking reciprocal relationship is crucial to the emergence of navigable small-world networks.

5. EMPIRICAL EVALUATION

In this section, we empirically examine the stability of navigable small-world NE. We simulate the DRB game on two dimensional grids, and consider nodes having full information, limited information, or no information of other players' strategies.

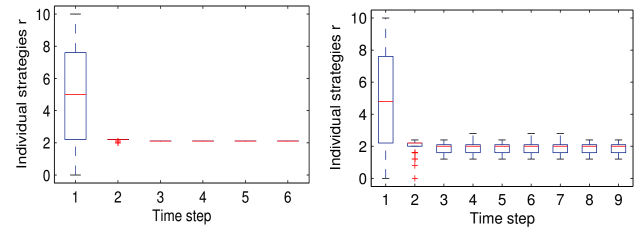
Before the main empirical evaluation, we also test the effect of the grid size on navigable equilibrium, since our theoretical results require sufficiently large grids. We found that for $\mathbf{r} \equiv 2.4$ for 10×10 grid and $\mathbf{r} \equiv 2.03$ for a 1000×1000 grid, with the granularity of $\gamma = 0.01$ (see [37] for more details). In our following experiments, we use a 100×100 grid with the navigable equilibrium $\mathbf{r} \equiv 2.1$.

5.1 Stability of NE under Perturbation

To demonstrate the stability of navigable NE, we simulate the DRB game with random perturbation. At time step 0, each player is perturbed independently with probability p. If the perturbation occurs on a player u, we assume that the player u chooses a new strategy uniformly at random from the interval $[0, 10] \cap \Sigma$. Notice that for strategy $r_u > 10$, the behavior of nodes is similar to $r_u = 10$ as nodes only connect to the 4 grid neighbors. We use a granularity of $\gamma = 0.1$ that covers the equilibrium $\mathbf{r} \equiv 2.1$ while reducing the simulation cost. Let $\mathbf{r}^0$ be the strategy profile at time 0 after the perturbation. At each time step $t \geq 1$, every player picks the best strategy based on the strategies of others in the previous step.

$$r_u^t = \operatorname*{argmax}_{r_u \in \Sigma \cap [0,10]} \pi(r_u, \mathbf{r}_{-u}^{t-1}), \forall u, \forall t > 1.$$

Figure 5 shows an extreme case where every player is perturbed when the initial profile is $\mathbf{r} \equiv 2$. The box-plot shows the distribution of players' strategies at each step. The figure shows that in just two steps the system returns to the navigable small-world NE. We tested 100 random starting profiles, and all of them converge to the navigable NE within two steps. This simulation result indicates that the navigable NE is very stable for random perturbations.



(a) Noise $\varepsilon = 0$. (b) Noise $\varepsilon \sim N(0, 0.5)$

Figure 7: Network evolution where each player only knows the strategies of their friends.

To contrast, we study the stability of the random small-world network in terms of tolerating perturbations. Figure 6 shows the result of randomly perturbing only 1% of players at the random NE, which are shown as the outliers at step 0. Note that 1% perturbation does not meet the requirement in Theorem 6. However, this small fraction of players would affect the decision of additional players in their vicinity, who can significantly improve the reciprocity by also linking in the vicinity. The figure clearly shows that in a few steps, more and more players would change their strategies, and the system finally goes to the navigable small-world NE.[2] We tested 100 random starting profiles, and all of them converge to the navigable NE within at most 12 steps.

These results show that the navigable small-world NE are robust to perturbations, while random small-world NE is not stable and easily transits to the small-world NE under a slight perturbation.

5.2 DRB Game with Limited Knowledge

Scenario 1: knowing friends' strategies. In practice, a player does not know the strategies of all players. So we now consider a weaker scenario where a player only knows the strategies of their friends. With these limited knowledge, a player can guess the strategies of all other players and pick the best response to the estimated strategies of all players.

To examine the convergence of navigable small-world NE in this scenario, we simulate the DRB game as follows. At time step 0, each player chooses an initial strategy uniformly at random from the interval $[0, 10] \cap \Sigma$. At every step $t \geq 0$, each player u creates q out-going long-range links based on her current strategy r_u^t, and learns the connection preferences of these q long-range contacts. Let F_u^t be the set of these q long-range contacts. We further allows a random noise term ε for each connection preference learned from the friends. Let r_v^t ($v \in F_u^t$) be the learned (noisy) connection preference. Then based on these newly learned connection preferences, player u estimates the strategies of all other players. One reasonable estimation method is to assume that players close to one another in grid distance have similar strategy. More specifically, for a non-friend n-

[2] In step 1 and 2 in Figure 6, the number of outliers is larger than in step 0, even though the rendering make it seems they are less.

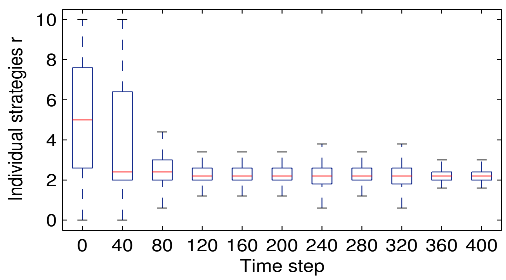


Figure 8: Network evolution where players have no knowledge of strategies of others.

ode $v \notin F_u^t$, u estimates the strategy of v by the average weight of known strategies:

$$r_v^t = \frac{\sum_{f \in F_u^t} r_{f,t-1}/d_M(v,f)}{\sum_{f \in F_u^t} 1/d_M(v,f)}.$$

Here we do not use the connection preferences learned in the previous steps and effectively assume that those old links are removed. This is both for convenience, and also reasonable since people could only maintain a limited number of connections and it is natural that new connections replace the old ones. Moreover, the connection preferences of those old connections may become out-dated in practice anyway. After the estimation procedure, player u use the strategy r_v^t from all other players (either learned or estimated) to compute its best response r_u^{t+1} for the next step.

In our experiment, we set $q = 30$. Figure 7(a) shows that when players have accurate knowledge of the strategies of their friends without noise, the system converges in just two steps. Even when the information on friends' strategies is noisy, the system can still quickly stabilize in a few steps to a state close to the navigable small-world NE, as shown in Figure 7(b). We tested 100 random starting profiles and also other estimation methods such as randomly choosing a connection preference based on friends' connection preference distributions, and results are all similar. This experiment further demonstrates the robustness of the small-world NE even under limited information on connection preferences.

Scenario 2: No information about others' strategies. We now consider the weakest scenario where each player has no knowledge about the strategies of other players. To get the payoff in this scenario, a player creates a certain number of links with the current strategy, and computes the payoff by multiplying the average link distance and the percentage of reciprocal links.

To make it even harder, we do not allow the player to try many different strategies at each step before fixing her strategy for the step. Instead, at each step each player only has one chance to slightly modify her current strategy. If the new strategy yields better payoff, the player would adopt the new strategy. So as the time goes on, the player could change the strategy towards the best one.

We simulate the DRB game as follows: At time step 0, each player chooses an initial strategy uniformly at random from the interval $[0, 10] \cap \Sigma$. Every player creates q outgoing links with her current strategy. At each time step $t \geq 1$, each player changes the strategy, i.e.,$r_u \leftarrow r_u + \delta$, and creates q new links with this new strategy, where δ is a random number determined as follows. First, for the sign of δ, in the first step it is randomly assigned positive or negative sign with equal probability; in the raining steps, to make the search efficient, we keep the sign of δ if the previous change leads to a higher payoff; otherwise we reverse the sign of δ. For the magnitude of δ, i.e. $|\delta|$, we sample a value uniformly at random from $(0, 1] \cap \Sigma$.

We simulate this system with $q = 30$. Figure 8 demonstrates that the system can still evolve to a state close to the navigable small-world NE in a few hundred steps, e.g., the strategies of 80.5% players fall in the interval $[1.8, 2.4]$, and the median of the strategies is the navigable NE strategy of 2.1. We test 50 random starting profiles, and take snapshots of the strategy profiles at the time step $t = 500$. On average, the strategies of 79.8% players in the snapshots fall in the interval $[1.8, 2.4]$.

In summary, our empirical evaluation strongly supports that our payoff function considering the balance between link distance and reciprocity naturally gives rise to the navigable small-world network. The convergence to navigable equilibrium will happen either when the players know all other players' strategies, or only learn their friends' strategies, or only use the empirical distance and reciprocity measure. Once in the navigable equilibrium, the system is very stable and hard to deviate by any random perturbation. Furthermore, other equilibria such as the random small world is not stable, in that a small perturbation will drive the system back to the navigable small-world network.

6. FUTURE WORK

Our study opens many possible directions of future work. For example, one may extend the current study to non-uniform population distributions on the grid and conduct theoretical and empirical analysis to see if the navigable small world would still emerge as a stable equilibrium. Another direction is to investigate deeper reasons or models on why individual's connection preference follows a power law form of $d_M(u,v)^{-r_u}$. Such studies may need to integrate prior studies on node and link dynamics and our game theoretic approach, and the integration may provide a more complete picture of the underlying mechanisms for navigable small-world networks.

7. ACKNOWLEDGMENTS

We thank the anonymous reviewers for their comments. This work was supported by the National Basic Research Program of China (Grant No. 2014CB340400), the National Science Foundation for Young Scholars of China (Grant No.61202423) and the Star Track Program of Microsoft Research Asia.

APPENDIX

A. PROOF OF THEOREM 1 FOR THE CASE OF $s > 0$

In all proofs in the appendix, for a given node $u \in V$, we denote $D(r_u) = \sum_{\forall v \neq u} p_u(v, r_u) d_M(u, v)$ as its average grid distance of its long range contacts (simply referred to as the *link distance*), and $P_u(r_u, \mathbf{r}_{-u}) = \sum_{\forall v \neq u} p_u(v, r_u) p_v(u, r_v)$ as its *reciprocity*. When $\mathbf{r}_{-u} \equiv s$, we simply use $P(r_u, s)$ to denote $P_u(r_u, \mathbf{r}_{-u} \equiv s)$. The subscript u in both $D(r_u)$ and $P(r_u, s)$ is omitted because their values are the same for all

$u \in V$. Let n_D be the longest grid distance among nodes in $K(n,k,p,q,\mathbf{r})$. We have that $n_D = k\lfloor n/2 \rfloor$. We denote $b_u(j)$ as the number of players at grid distance j from u. We can find two constants ξ_k^- and ξ_k^+ only depending on the dimension k, so that $\xi_k^- j^{k-1} \le b_u(j) \le \xi_k^+ j^{k-1}$ for $1 \le j \le \lfloor n/2 \rfloor$ and $1 \le b_u(j) \le \xi_k^+ j^{k-1}$ for $\lfloor n/2 \rfloor < j \le n_D$.[3] Note that the payoff function (Eq. (3)) for the DRB game is indifferent of parameters p and q of the network, so we treat $p = q = 1$ for our convenience in the analysis.

Proof of $B_u(\mathbf{r}_{-u} \equiv s) = k$ if $s > 0$. Given $\mathbf{r_{-u}} \equiv s$, we now prove that the decision of $r_u = k$ is the unique best response of player u as long as $s > 0$.

Payoff of $r_u = k$: We have the lower bound for the link distance and the reciprocity:

$$D(r_u) \ge \frac{\sum_{j=1}^{n/2} b_u(j)\cdot j^{-k}\cdot j}{c(k)} \ge \frac{\xi_k^- n}{2c(k)}, \tag{4}$$

$$P(r_u,s) \ge \frac{\sum_{j=1}^{n/2} b_u(j)\cdot j^{-k}\cdot j^{-s}}{c(k)c(s)} \ge \xi_k^- \sum_{j=1}^{n/2} \frac{j^{-1-s}}{c(k)c(s)} \ge \frac{\xi_k^-}{c(k)c(s)}. \tag{5}$$

Notice that the normalization coefficient $c(k)$ can be bounded as:

$$c(k) = \sum_{\forall v \ne u} d_M(u,v)^{-k} \le \sum_{j=1}^{n_D} b_u(j) j^{-k} \le \xi_k^+ \sum_{j=1}^{n_D} \frac{1}{j} \le \xi_k^+ \ln(2kn). \tag{6}$$

Based on above inequalities, we can get the lower bound for the payoff of player u in the case of $r_u = k$:

$$\pi(r_u = k, \mathbf{r}_{-u} \equiv s) \ge \frac{(\xi_k^-)^2}{2(\xi_k^+)^2 c(s)} \frac{n}{\ln^2(2kn)}. \tag{7}$$

Payoff of $r_u < k$. We now turn to the player's payoff when $r_u < k$ and we write $\varepsilon = k - r_u$ ($\gamma \le \varepsilon \le k$). In this case, the upper bound for the link distance and reciprocity are as follows:

$$\begin{aligned} D(r_u) &\le \frac{\sum_{j=1}^{n_D} b_u(j)\cdot j^{-r_u}\cdot j}{c(r_u)} \le \frac{\xi_k^+ \int_1^{n_D+1} x^{\varepsilon} dx}{c(r_u)} \\ &\le \frac{\xi_k^+ (n_D+1)^{1+\varepsilon}}{(1+\varepsilon)c(r_u)} \le \frac{\xi_k^+ (kn)^{1+\varepsilon}}{c(r_u)}, \end{aligned} \tag{8}$$

$$P(r_u,s) \le \frac{\sum_{j=1}^{n_D} b_u(j)\cdot j^{-r_u}\cdot j^{-s}}{c(r_u)c(s)} \le \xi_k^+ \sum_{j=1}^{n_D} \frac{j^{\varepsilon-1-s}}{c(r_u)c(s)}. \tag{9}$$

In the case of $\varepsilon \ge 1+s$, we have:

$$P(r_u,s) \le \xi_k^+ \int_{j=1}^{n_D+1} \frac{j^{\varepsilon-1-s}}{c(r_u)c(s)} \le \frac{\xi_k (n_D+1)^{\varepsilon-s}}{(\varepsilon-s)c(r_u)c(s)} \le \frac{\xi_k^+ (kn)^{\varepsilon-s}}{c(r_u)c(s)}. \tag{10}$$

In the case of $\varepsilon < 1+s$, we have:

$$\begin{aligned} P(r_u,s) &\le \frac{\xi_k^+}{c(r_u)c(s)} \left(1 + \int_{j=1}^{n_D} j^{\varepsilon-1-s}\right) \\ &\le \begin{cases} \frac{\xi_k^+((kn/2)^{\varepsilon-s}+\varepsilon-s-1)}{(\varepsilon-s)c(r_u)c(s)} & \text{if } \varepsilon \ne s, \\ \frac{2\xi_k^+ \ln(2kn)}{c(r_u)c(s)} & \text{if } \varepsilon = s. \end{cases} \end{aligned} \tag{11}$$

[3]The exact values of ξ_k^- and ξ_k^+ can be derived by the combinatorial problem of counting the number of ways to choose k non-negative integers such that they sum to a given positive integer j.

The coefficient $c(r_u)$ can be bounded as:

$$\begin{aligned} c(r_u) &= \sum_{\forall v \ne u} d_M(u,v)^{-r_u} \ge \sum_{j=1}^{n/2} b_u(j) j^{-r_u} \ge \xi_k^- \sum_{j=1}^{n/2} j^{\varepsilon-1} \\ &\ge \xi_k^- \int_1^{n/2} x^{\varepsilon-1} dx \ge \frac{\xi_k^-}{\varepsilon}\left(\frac{n}{2}\right)^{\varepsilon} - \frac{\xi_k^-}{\varepsilon} \ge \frac{\xi_k^-}{2\varepsilon}\left(\frac{n}{2}\right)^{\varepsilon}. \end{aligned} \tag{12}$$

The last inequality above relies on a loose relaxation of $\frac{1}{2}\left(\frac{n}{2}\right)^{\varepsilon} \ge 1$, which is guaranteed for all $n \ge 2^{1+1/\gamma}$ since $\varepsilon \ge \gamma$.

Combining Eq.(8), (10), (11), and (12), we have the payoff of node u in the case of $r_u < k$:

$$\pi(r_u = k-\varepsilon, \mathbf{r}_{-u} \equiv s) \le$$

$$\begin{cases} \text{if } \varepsilon > s, \\ \frac{(\xi_k^+)^2 2^{2\varepsilon+2}\varepsilon^2 k^{1+2\varepsilon-s}}{(\xi_k^-)^2(\varepsilon-s)c(s)} n^{1-s} \le \frac{(\xi_k^+)^2 2^{2k+2} k^{2k+3}}{(\xi_k^-)^2 \gamma c(s)} n^{1-\gamma}, \\ \text{if } \varepsilon = s, \\ \frac{(\xi_k^+)^2 2^{2\varepsilon+3} 2\varepsilon^2 k^{1+\varepsilon} n^{1-\varepsilon} \ln(2kn)}{(\xi_k^-)^2 c(s)} \le \frac{(\xi_k^+)^2 2^{2k+3} k^{k+3}}{(\xi_k^-)^2 c(s)} n^{1-\gamma} \ln(2kn), \\ \text{if } \varepsilon < s, \\ \frac{(\xi_k^+)^2 2^{2\varepsilon+2}\varepsilon^2 k^{1+\varepsilon}}{(\xi_k^-)^2 c(s)} \left(1 + \frac{1}{s-\varepsilon}\right) n^{1-\varepsilon} \le \frac{(\xi_k^+)^2 2^{2k+3} k^{k+3}}{(\xi_k^-)^2 \gamma c(s)} n^{1-\gamma}. \end{cases} \tag{13}$$

The inequalities in the three cases above use the facts $\gamma \le \varepsilon \le k$, $s \ge \gamma$ (since $s > 0$), and $\varepsilon - s \ge \gamma$ when $\varepsilon > s$.

Comparing Eq.(13) with Eq.(7), the common denominator $c(s)$ can be ignored. For the rest terms, Eq.(7) is in strictly higher order in n than Eq.(13), which implies that we can find a large enough n_0 (only depending on model constants k and γ) such that for all $n \ge n_0$, $\pi(r_u = k, \mathbf{r}_{-u} \equiv s) > \pi(r_u = s', \mathbf{r}_{-u} \equiv s)$, for all $s \in \Sigma \setminus \{0\}$ and all $s' \in \Sigma$ with $s' < k$.

Payoff of $r_u > k$. We write $\varepsilon = r_u - k$ ($\varepsilon \ge \gamma$). The bound for the link distance is:

$$\begin{aligned} D(r_u) &\le \frac{\sum_{j=1}^{n_D} b_u(j)\cdot j^{-r_u}\cdot j}{c(r_u)} \le \xi_k^+ \sum_{j=1}^{n_D} \frac{j^{-\varepsilon}}{c(r_u)} \\ &\le \xi_k^+ \frac{1 + \int_1^{n_D} x^{-\varepsilon} dx}{c(r_u)} \le \begin{cases} \frac{\xi_k^+}{(1-\varepsilon)c(r_u)} (kn/2)^{1-\varepsilon} & \text{if } \varepsilon < 1, \\ \frac{\xi_k^+}{c(r_u)} \ln(2kn) & \text{if } \varepsilon \ge 1. \end{cases} \end{aligned} \tag{14}$$

The bound on the reciprocity is:

$$P(r_u,s) \le \frac{\sum_{j=1}^{n_D} b_u(j)\cdot j^{-r_u}\cdot j^{-s}}{c(r_u)c(s)} \le \frac{\sum_{j=1}^{n_D} b_u(j) j^{-r_u}}{c(r_u)c(s)} = \frac{1}{c(s)}. \tag{15}$$

On the other hand, the lower bound for the coefficient $c(r_u)$ is:

$$c(r_u) = \sum_{\forall v \ne u} d_M(u,v)^{-r_u} \ge \sum_{j=1}^{n/2} b_u(j) j^{-r} \ge b_u(1) \ge \xi_k^-. \tag{16}$$

Combining Eq.(14), (15), and (16), we get:

$$\pi(r_u = k+\varepsilon, \mathbf{r_{-u}} \equiv s) \le$$

$$\begin{cases} \frac{\xi_k^+ k^{1-\varepsilon}}{\xi_k^-(1-\varepsilon)c(s)} n^{1-\varepsilon} \le \frac{\xi_k^+ k}{\xi_k^- \gamma c(s)} n^{1-\gamma} & \text{if } \varepsilon < 1, \\ \frac{\xi_k^+}{\xi_k^- c(s)} \ln(2kn) & \text{if } \varepsilon \ge 1. \end{cases} \tag{17}$$

Comparing Eq.(17) with Eq.(7), the common denominator $c(s)$ can be ignored. For the rest terms, Eq.(7) is in strictly higher order in n than Eq.(17), which implies that we can find a large enough constant n_0 (only depending on model constants k and γ) such that for all $n \geq n_0$, $\pi(r_u = k, \mathbf{r}_{-u} \equiv s) > \pi(r_u = s', \mathbf{r}_{-u} \equiv s)$, for all $s \in \Sigma \setminus \{0\}$ and all $s' \in \Sigma$ with $s' > k$.

We complete the proof when combining the cases of $r_u < k$ and $r_u > k$. $\square$

B. PROOF OF THEOREM 3

We introduce some notations first. Given the strategy profile $\mathbf{r}$ and a node u with $r_u \neq k$, we partition the rest nodes $V \setminus \{u\}$ into three sets: $V_{<k} = \{u \in V \setminus \{u\} \mid r_u < k\}$, $V_{>k} = \{u \in V \setminus \{u\} \mid r_u > k\}$, $V_{=k} = \{u \in V \mid r_u = k\}$. For any $A \subseteq V$, Let $P_{u,A}(\mathbf{r}) = \sum_{v \in A} p_u(v, r_u) p_v(u, r_v)$ be the reciprocity u obtained from subset A. Then we have

$$\pi_u(\mathbf{r}) = D(r_u)\left(P_{u,V_{<k}}(\mathbf{r}) + P_{u,V_{>k}}(\mathbf{r}) + P_{u,V_{=k}}(\mathbf{r})\right). \qquad (18)$$

In this section, we actually prove a slightly stronger result: any node u in any strategy profile $\mathbf{r}$ with $r_u \neq k$ is strictly worse off than its payoff in the navigable equilibrium, when n is large enough. To prove this result, we first show the following key lemma, which will be used in later theorems too.

LEMMA 1. *In the k-dimensional DRB game, there exists a constant κ (only depending on model constants k and γ), for sufficiently large n (in particular $n \geq \max(e^4, 2k)$), the following statement holds: for any strategy profile $\mathbf{r}$, any node u with $r_u \neq k$, $\pi_u(r_u, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$.*

PROOF. We now consider the case of $r_u < k$ and $r_u > k$ separately.

Payoff of $r_u < k$. Let $\varepsilon = k - r_u$ ($\gamma \leq \varepsilon \leq k$). We first consider the reciprocity player u obtains from the players in $V_{<k}$. We have $c(r_v) \geq c(k-\gamma)$ for $\forall v \in V_{<k}$, since $r_v \leq k-\gamma$. Then we have:

$$P_{u,V_{<k}}(\mathbf{r}) = \sum_{v \in V_{<k}} \frac{d_M(u,v)^{-r_u - r_v}}{c(r_u)c(r_v)} \leq \sum_{v \in V_{<k}} \frac{d_M(u,v)^{-r_u}}{c(r_u)c(k-\gamma)} \leq \frac{\sum_{\forall v \neq u} d_M(u,v)^{-r_u}}{c(r_u)c(k-\gamma)} = \frac{1}{c(k-\gamma)}. \qquad (19)$$

Combining the above inequality with the bounds in Eq.(8) and (12), we get:

$$D(r_u)P_{u,V_{<k}}(\mathbf{r}) \leq \frac{\xi_k^+ 2^{\varepsilon+3} \varepsilon k^{1+\varepsilon}}{(\xi_k^-)^2} n^{1-\gamma} \leq \frac{\xi_k^+ 2^{k+3} k^{k+2}}{(\xi_k^-)^2} n^{1-\gamma}. \qquad (20)$$

Next we examine the reciprocity that player u obtains from the players in $V_{>k}$. Note that for all $v \in V_{>k}$, $r_v \geq k + \gamma$. Using Eq.(16), we have:

$$P_{u,V_{>k}}(\mathbf{r}) = \sum_{v \in V_{<k}} \frac{d_M(u,v)^{-r_u - r_v}}{c(r_u)c(r_v)} \leq \frac{\sum_{j=1}^n b_u(j) \cdot j^{-r_u} \cdot j^{-k-\gamma}}{\xi_k^- c(r_u)} = \frac{\xi_k^+ \sum_{j=1}^n j^{-1-r_u-\gamma}}{\xi_k^- c(r_u)} \leq \frac{\xi_k^+ (1 + \int_1^n x^{-1-r_u-\gamma} dx)}{\xi_k^- c(r_u)} \leq \frac{\xi_k^+ (1 + r_u + \gamma)}{\xi^- (r_u + \gamma) c(r_u)} \leq \frac{\xi^+ (k+1)}{\xi^- \gamma c(r_u)}.$$

Based on the bounds in Eq.(8) and (12), we get:

$$D(r_u)P_{u,V_{>k}}(\mathbf{r}) \leq \frac{\xi_k^+ 2^{2\varepsilon+2} \varepsilon^2 (k+1) k^{1+\varepsilon}}{(\xi_k^-)^2 \gamma} n^{1-\varepsilon} \leq \frac{\xi_k^+ 2^{2k+3} k^{k+4}}{(\xi_k^-)^2 \gamma} n^{1-\gamma}. \qquad (21)$$

We now examine the reciprocity of player u from players in $V_{=k}$. Notice that the coefficient $c(k)$ can be bounded as:

$$c(k) \geq \xi_k \sum_{j=1}^{n/2} j^{-1} \geq \xi_k^- \int_1^{n/2} x^{-1} dx \geq \xi_k^- (\ln n - \ln 2) \geq \frac{\xi_k^- \ln n}{2}, \qquad (22)$$

where the last inequality is true when $n \geq e^4$. Using Eq.(13) with $s = k$, together with Eq.(22), we get:

$$D(r_u)P_{u,V_{=k}}(\mathbf{r}) \leq \pi(r_u = k - \varepsilon, \mathbf{r}_{-u} \equiv k) \leq \begin{cases} \frac{(\xi_k^+)^2 2^{2k+4} k^{k+3} \ln(2kn)}{(\xi_k^-)^3 \ln n} n^{1-\gamma} \leq \frac{(\xi_k^+)^2 2^{2k+5} k^{k+3}}{(\xi_k^-)^3} n^{1-\gamma} & \text{if } \varepsilon = k. \\ \frac{(\xi_k^+)^2 2^{2k+4} k^{k+3}}{\gamma (\xi_k^-)^3} \frac{n^{1-\gamma}}{\ln n} & \text{if } \varepsilon < k. \end{cases} \qquad (23)$$

The last inequality in the case of $\varepsilon = k$ requires $n \geq 2k$.

Adding up results in Eq.(20), (21), (23), we obtain that

$$\pi(r_u, \mathbf{r}_{-u}) \leq \frac{3(\xi_k^+)^2 \cdot 2^{2k+5} k^{k+4}}{\gamma (\xi_k^-)^3} n^{1-\gamma} \leq \frac{(\xi_k^+)^2 2^{2k+7} k^{k+4}}{\gamma (\xi_k^-)^3} n^{1-\gamma}, \qquad (24)$$

when $n \geq \max\{e^4, 2k\}$.

Payoff of $r_u > k$. Let $\varepsilon = r_u - k$ ($\varepsilon \geq \gamma$). For this case, we can relax the reciprocity $P_u(r_u, \mathbf{r}_{-u})$ to one and only upper bound link distance $D(r_u)$. Applying bounds in Eq.(14) and (16), we obtain:

$$\pi(r_u = k + \varepsilon, \mathbf{r}_{-u}) \leq \begin{cases} \frac{\xi_k^+}{\xi_k^- (1-\varepsilon)} (kn/2)^{1-\varepsilon} \leq \frac{\xi_k^+ k}{\xi_k^- \gamma} n^{1-\gamma} & \text{if } \varepsilon < 1, \\ \frac{\xi_k^+}{\xi_k^-} \ln(2kn) \leq 2n^{1-\gamma} & \text{if } \varepsilon \geq 1. \end{cases} \qquad (25)$$

The last inequality in the above case of $\varepsilon \geq 1$ holds when $n \geq 2k$ and $\gamma \leq 1/2$.

Finally, the lemma holds when we combine Eq.(24) and (25) $\square$

PROOF OF THEOREM 3. Before derivation, a player u has the following payoff lower bound, according to Eq.(6) and (7).

$$\pi(r_u = k, \mathbf{r}_{-u} \equiv k) \geq \frac{(\xi_k^-)^2}{2(\xi_k^+)^3} \frac{n}{\ln^3(2kn)}. \qquad (26)$$

Suppose that a coalition C deviates, and the new strategy profile is $\mathbf{r}$. Then some node $u \in C$ must select a new $r_u \neq k$. By Lemma 1, there is a constant κ such that for all sufficiently large n, $\pi(r_u, \mathbf{r}_{-u}) \leq \kappa n^{1-\gamma}$. Comparing with Eq. (26), we see that the payoff of u before the deviation is in strictly higher order in n than its payoff after the deviation. Therefore, for all sufficiently large n, u is strictly worse off, which means no coalition could make some member strictly better off while others not worse off. Hence, navigable small-world network ($\mathbf{r} \equiv k$) is a strong Nash equilibrium. $\square$

C. REFERENCES

[1] Adamic, L., and Adar, E. How to search a social network. *Social Networks 27* (2005).

[2] Allamanis, M., Scellato, S., and Mascolo, C. Evolution of a location-based online social network: Analysis and models. In *Proc. of IMC* (2012).

[3] Barabási, A., and Albert, R. Emergence of scaling in random networks. *Science 286*, 5439 (1999), 509–512.

[4] Boguñá, M., Krioukov, D., and Claffy, K. Navigability of complex networks. *Nature Physics 5* (2009), 74–80.

[5] Chaintreau, A., Fraigniaud, P., and Lebhar, E. Networks become navigable as nodes move and forget. In *Proc. of ICALP '08* (2008).

[6] Chen, W., Fang, W., Hu, G., and Mahoney, M. W. On the hyperbolicity of small-world and treelike random graphs. *Internet Mathematics 9*, 4 (2013), 434–491.

[7] Cho, E., Myers, S. A., and Leskovec, J. Friendship and mobility: User movement in location-based social networks. In *Proc. of KDD* (2011).

[8] Clauset, A., and Moore, C. How do networks become navigable. In *Arxiv preprint arXiv:0309.415v2* (2003).

[9] Gittell, R., and Vidal, A. *Community Organizing: Building Social Capital as a Development Strategy.* Sage publicaiton, 1998.

[10] Goldenberg, J., and Levy, M. Distance is not dead: Social interaction and geographical distance in the internet era. In *Arxiv preprint arXiv:0906.3202.* (2009).

[11] Gong, N. Z., Xu, W., Huang, L., Mittal, P., Stefanov, E., Sekar, V., and Song, D. Evolution of social-attribute networks: Measurements, modeling, and implications using google+. In *Proc. of IMC* (2012).

[12] Gouldner, A. W. The norm of reciprocity: a preliminary statement. *American Sociological Review 25*, 4 (1960), 161–178.

[13] Granovetter, M. *Getting a job: A study of contacts and careers.* Harvard University Press, 1974.

[14] Granovetter, M. S. The strength of weak ties. *American Journal of Sociology* (1973), 1360–1380.

[15] Gulyás, A., Korösi, A., Szabó, D., and Biczóki, G. On greedy network formation. *SIGMETRICS Perform. Eval. Rev.* (2012).

[16] Hu, Y., Wang, Y., Li, D., Havlin, S., , and Di, Z. Possible origin of efficient navigation in small worlds. *Phys Rev Letters* (2011).

[17] Illenberger, J., Nagel, K., and Flötteröd, G. The role of spatial interaction in social networks. *Networks and Spatial Economics 13* (2013).

[18] Java, A., Song, X., Finin, T., and Tseng, B. Why we twitter: understanding microblogging usage and communities. In *Proc. of WebKDD/SNA-KDD* (2007).

[19] Jiang, J., Wilson, C., Wang, X., Huang, P., Sha, W., Dai, Y., and Zhao, B. Y. Understanding latent interactions in online social networks. In *Proc. of IMC* (2010).

[20] Kaiser, M., and Hilgetag, C. C. Spatial growth of real-world networks. *Phys Rev Letters* (2004).

[21] Kleinberg, J. The small-world phenomenon: An algorithmic perspective. In *Proc. of 32nd ACM Symp. on Theory of Computing (STOC)* (2002).

[22] Kleinberg, J. Complex networks and decentralized search algorithms. In *Proc. of the International Congress of Mathematicians (ICM)* (2006).

[23] Krioukov, D., Papadopoulos, F., Kitsak, M., Vahdat, A., and Boguñá, M. Hyperbolic geometry of complex networks. *Physical Review E 82* (2010), 036106.

[24] Krioukov, D., Papadopoulos, F., Vahdat, A., and Boguñá, M. Curvature and temperature of complex networks. *Physical Review E 80* (2009), 035101(R).

[25] Lambiotte, R., et al. Geographical dispersal of mobile communication networks. *Physica A 387* (2008).

[26] Leskovec, J., Backstrom, L., Kumar, R., and Tomkins, A. Microscopic evolution of social networks. In *Proc. of KDD* (2008).

[27] Leskovec, J., Kleinberg, J. M., and Faloutsos, C. Graph evolution: Densification and shrinking diameters. *TKDD* (2007).

[28] Liben-Nowell, D., Novak, J., Kumar, R., Raghavan, P., and Tomkins, A. Geographic routing in social networks. *PNAS 102* (2005).

[29] Mathias, N., and Gopal, V. Small worlds: how and why. *Physical Review E 63*, 2 (2001).

[30] Milgram, S. The small world problem. *Psychology Today 2*, 1 (1967), 60–67.

[31] Mislove, A., Marcon, M., Gummadi, K. P., Druschel, P., and Bhattacharjee, B. Measurement and analysis of online social networks. In *Proc. of IMC* (2007).

[32] Papadopoulos, F., Krioukov, D. V., Boguñá, M., and Vahdat, A. Greedy forwarding in dynamic scale-free networks embedded in hyperbolic metric spaces. In *Proceedings of the 29th IEEE International Conference on Computer Communications (INFOCOM)* (2010), pp. 2973–2981.

[33] Sandberg, O., and Clarke, I. The evolution of navigable small-world networks. In *Arxiv preprint arXiv:cs/0607025* (2006).

[34] Schaller, M., and Latank, B. Distance matters: Physical space and social impact. *Personality and Social Psychology Bulletin 25* (1995).

[35] Tardos, E., and Wexler, T. Network formation games and the potential function method, 2007. In Algorithmic Game Theory.

[36] Watts, D. J., and Strogatz, S. H. Collective dynamics of 'small-world' networks. *Nature 393*, 6684 (1998), 440–442.

[37] Yang, Z., and Chen, W. A game theoretic model for the formation of navigable small-world networks, 2014. full technical report, Arxiv preprint arXiv:1411.4097.

[38] Yang, Z., Wilson, C., Wang, X., Gao, T., Zhao, B. Y., and Dai, Y. Uncovering social network sybils in the wild. In *Proc. of IMC* (2011).